Bubbling surface operators and S-duality

Jaume Gomis[a] **and Shunji Matsuura**[ab]

[a]*Perimeter Institute for Theoretical Physics*
Waterloo, Ontario N2L 2Y5, Canada
[b]*Department of Physics, University of Tokyo,*
7-3-1 Hongo, Tokyo
E-mail: jgomis@perimeterinstitute.ca, smatsuura@perimeterinstitute.ca

ABSTRACT: We construct smooth asymptotically $AdS_5 \times S^5$ solutions of Type IIB supergravity corresponding to all the half-BPS surface operators in $\mathcal{N} = 4$ SYM. All the parameters labeling a half-BPS surface operator are identified in the corresponding bubbling geometry. We use the supergravity description of surface operators to study the action of the $SL(2, Z)$ duality group of $\mathcal{N} = 4$ SYM on the parameters of the surface operator, and find that it coincides with the recent proposal by Gukov and Witten in the framework of the gauge theory approach to the geometrical Langlands with ramification. We also show that whenever a bubbling geometry becomes singular that the path integral description of the corresponding surface operator also becomes singular.

Contents

1. Introduction and summary

Gauge invariant operators play a central role in the gauge theory holographically describing quantum gravity with AdS boundary conditions $[1-3]$, as correlation functions of gauge invariant operators are the only observables in the boundary gauge theory. Finding the bulk description of all gauge invariant operators is necessary in order to be able to formulate an arbitrary bulk experiment in terms of gauge theory variables.

In this paper we provide the bulk description of a novel class of half-BPS operators in $\mathcal{N} = 4$ SYM which are supported on a surface Σ [4]. These nonlocal surface operators $\mathcal{O}_\Sigma$ are defined by quantizing $\mathcal{N} = 4$ SYM in the presence of a certain codimension two singularity for the classical fields of $\mathcal{N} = 4$ SYM. The singularity characterizing such a surface operator $\mathcal{O}_\Sigma$ depends on $4M$ real parameters, where M is the number of $\mathrm{U}(1)$'s left unbroken by $\mathcal{O}_\Sigma$. Surface operators are a higher dimensional generalization of Wilson and 't Hooft operators, which are supported on curves and induce a codimension three singularity for the classical fields appearing in the Lagrangian. In this paper we extend the bulk description of all half-BPS Wilson loop operators found in [5] (see also[1] $[8-12]$) to all half-BPS surface operators.

We find the asymptotically $\mathrm{AdS}_5 \times \mathrm{S}^5$ solutions of Type IIB supergravity corresponding to all half-BPS surface operators $\mathcal{O}_\Sigma$ in $\mathcal{N} = 4$ $\mathrm{U}(N)$ SYM. The topology and geometry of the "bubbling" solution is completely determined in terms of some data, very much like in the case studied by Lin, Lunin and Maldacena (LLM) in the context of half-BPS local operators [13].[2] In fact, we identify the system of equations determining the supergravity solution corresponding to the half-BPS surface operators in $\mathcal{N} = 4$ SYM with that obtained by "analytic" continuation of the LLM equations [13, 16].

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[1]The description of Wilson loops in the fundamental representation goes back to [6, 7].

[2]The bubbling geometry description of half-BPS Wilson loops was found in [10, 11] while that of half-BPS domain wall operators was found in [11, 14]. For the bubbling Calabi-Yau geometries for Wilson loops in Chern-Simons, see [15].
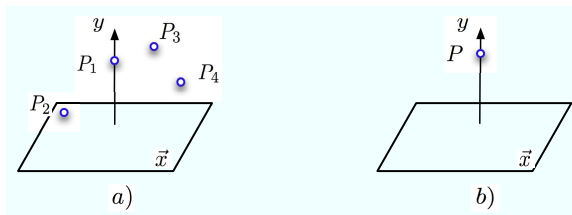
Figure 1: *a*) The metric and five-form flux is determined once the position of the particles in X – labeled by coordinates $(\vec{x}_l, y_l)$ where $y \geq 0$ — is given. The l-th particle is associated with a point $P_l \in X$. *b*) The configuration corresponding to the AdS$_5\times$S^5 vacuum.

The data determining the topology and geometry of a supergravity solution is characterized by the position of a collection of M point particles in a three dimensional space X, where X is a submanifold of the ten dimensional geometry. Different particle configurations give rise to different asymptotically AdS$_5\times$S^5 geometries.

Even though the choice of a particle distribution in X completely determines the geometry and topology of the metric and the corresponding RR five-form field strength, further choices have to be made to fully characterize a solution of Type IIB supergravity on this geometry.[3] Given a configuration of M particles in X, the corresponding ten dimensional geometry develops M non-trivial disks which end on the boundary[4] of AdS$_5\times$S^5 on a non-contractible S^1. Since Type IIB supergravity has two two-form gauge fields, one from the NS-NS sector and one from the RR sector, a solution of the Type IIB supergravity equations of motion is fully determined only once the holonomy of the two-forms around the various disks is specified:

$$\int_{D_l} \frac{B_{NS}}{2\pi} \qquad \int_{D_l} \frac{B_R}{2\pi} \qquad l = 1, \ldots, M. \qquad (1.1)$$

Therefore, an asymptotically AdS$_5\times$S^5 solution depends on the position of the M particles in X — given by $(\vec{x}_l, y_l)$ – and on the holonomies of the two-forms (1.1).

A precise dictionary is given between all the $4M$ parameters that label a half-BPS surface operator $\mathcal{O}_\Sigma$ and all the parameters describing the corresponding supergravity solution. We show that the supergravity solution describing a half-BPS surface operator is regular and that whenever the supergravity solution develops a singularity the $\mathcal{N} = 4$ SYM path integral description of the corresponding surface operator also develops a singularity.

We study the action of the SL$(2, Z)$ symmetry of Type IIB string theory on the supergravity solutions representing the half-BPS surface operators in $\mathcal{N} = 4$ SYM. By using the proposed dictionary between the parameters of a supergravity solution and the parameters of the corresponding surface operator, we can show that the action of S-duality induced on

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[3]This is on top of the obvious choice of dilaton and axion, which gets identified with the complexified coupling constant in $\mathcal{N} = 4$ SYM.

[4]The conformal boundary in this case is AdS$_3\times$S^1, where surface operators in $\mathcal{N} = 4$ SYM can be studied by specifying non-trivial boundary conditions.

the parameters of a surface operator coincides with the recent proposal by Gukov and Witten [4] in the framework of the gauge theory approach to the geometrical Langlands [17][5] with ramification.

Whether surface operators can serve as novel order parameters in gauge theory remains an important open question. It is our hope that the viewpoint on these operators provided by the supergravity solutions in this paper may help shed light on this crucial question.

The plan of the rest of the paper is as follows. In section 2 we study the gauge theory singularities corresponding to surface operators in $\mathcal{N} = 4$ SYM, study the symmetries preserved by a half-BPS surface operator and review the proposal in [4] for the action of S-duality on the parameters that a half-BPS surface operator depends on. We also compute the scaling weight of these operators and show that it is invariant under Montonen-Olive duality. In section 3 we construct the solutions of Type IIB supergravity describing the half-BPS surface operators. We identify all the parameters that a surface operator depends on in the supergravity solution and show that the action of S-duality on surface operators proposed in [4] follows from the action of $\mathrm{SL}(2, Z)$ on the classical solutions of supergravity. The appendices contain some details omitted in the main text.

2. Surface operators in gauge theories

A surface operator $\mathcal{O}_\Sigma$ is labeled by a surface Σ in $R^{1,3}$ and by a conjugacy class U of the gauge group G. The data that characterizes a surface operator $\mathcal{O}_\Sigma$, the surface Σ and the conjugacy class U, can be identified with that of an external string used to probe the theory. The surface Σ corresponds to the worldsheet of a string while the conjugacy class U is associated to the Aharonov-Bohm phase acquired by a charged particle encircling the string.

The singularity[6] in the gauge field produced by a surface operator is that of a non-abelian vortex. This singularity in the gauge field can be characterized by the phase acquired by a charged particle circumnavigating around the string. This gives rise to a group element[7] $U \subset \mathrm{U}(N)$

$$U \equiv P \exp i \oint A \subset \mathrm{U}(N), \tag{2.1}$$

which corresponds to the Aharonov-Bohm phase picked up by the wavefunction of the charged particle. Since gauge transformations act by conjugation $U \to gUg^{-1}$, a surface operator is labeled by a conjugacy class of the gauge group.

By performing a gauge transformation, the matrix U can be diagonalized. If we demand that the gauge field configuration is scale invariant — so that $\mathcal{O}_\Sigma$ has a well defined scaling

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[5]See e.g. [18] for a review of the geometric Langlands program.

[6]Previous work involving codimension two singularities in gauge theory include [19 − 21].

[7]We now focus on $G = \mathrm{U}(N)$ as it is the relevant gauge group for describing string theory with asymptotically $\mathrm{AdS}_5 \times \mathrm{S}^5$ boundary conditions.

weight — then the gauge field produced by a surface operator can then be written as

$$A = \begin{pmatrix} \alpha_1 \otimes 1_{N_1} & 0 & \ldots & 0 \\ 0 & \alpha_2 \otimes 1_{N_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \alpha_M \otimes 1_{N_M} \end{pmatrix} d\theta, \qquad (2.2)$$

where θ is the polar angle in the $R^2 \subset R^{1,3}$ plane normal to Σ and 1_n is the n-dimensional unit matrix. We note that the matrix U takes values on the maximal torus $T^N = R^N/Z^N$ of the $U(N)$ gauge group. Therefore the parameters α_i take values on a circle of unit radius.

The surface operator corresponding to (2.2) spontaneously breaks the $U(N)$ gauge symmetry along Σ down to the so called Levi group L, where a group of Levi type is characterized by the subgroup of $U(N)$ that commutes with (2.2). Therefore, $L = \prod_{l=1}^{M} U(N_l)$, where $N = \sum_{l=1}^{M} N_l$.

Since the gauge group is broken down to the Levi group $L = \prod_{l=1}^{M} U(N_l)$ along Σ, there is a further choice [4] in the definition of $\mathcal{O}_\Sigma$ consistent with the symmetries and equations of motion. This corresponds to turning on a two dimensional θ-angle for the unbroken $U(1)$'s along the string worldsheet Σ. The associated operator insertion into the $\mathcal{N} = 4$ SYM path integral is given by:

$$\exp\left(\sum_{l=1}^{M} \eta_l \int_\Sigma \text{Tr}\, \frac{F_l}{2\pi}\right). \qquad (2.3)$$

The parameters η_i takes values in the maximal torus of the S-dual or Langlands dual gauge group $^L G$ [4]. Therefore, since $^L G = U(N)$ for $G = U(N)$, we have that the matrix of θ-angles of a surface operator $\mathcal{O}_\Sigma$ characterized by the Levi group $L = \prod_{l=1}^{M} U(N_l)$ is given by the L-invariant matrix:

$$\eta = \begin{pmatrix} \eta_1 \otimes 1_{N_1} & 0 & \ldots & 0 \\ 0 & \eta_2 \otimes 1_{N_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \eta_M \otimes 1_{N_M} \end{pmatrix}. \qquad (2.4)$$

The parameters η_i, being two dimensional θ-angles, also take values on a circle of unit radius.

Therefore, a surface operator $\mathcal{O}_\Sigma$ in pure gauge theory with Levi group $L = \prod_{l=1}^{M} U(N_l)$ is labeled by $2M$ L-invariant parameters (α_l, η_l) up to the action of S_M, which acts by permuting the different eigenvalues in (2.2) and (2.4). The operator is then defined by expanding the path integral with the insertion of the operator (2.3) around the singularity (2.2), and by integrating over connections that are smooth near Σ. In performing the path integral, we must divide [4] by the gauge transformations that take values in $L = \prod_{l=1}^{M} U(N_l)$ when restricted to Σ. This means that the operator becomes singular whenever the unbroken gauge symmetry near Σ gets enhanced, corresponding to when eigenvalues in (2.2) and (2.4) coincide.

Surface operators in $\mathcal{N} = 4$ SYM. In a gauge theory with extra classical fields like $\mathcal{N} = 4$ SYM, the surface operator $\mathcal{O}_\Sigma$ may produce a singularity for the extra fields near the location of the surface operator. The only requirement is that the singular field configuration solves the equations of motion of the theory away[8] from the surface Σ. The global symmetries imposed on the operator $\mathcal{O}_\Sigma$ determine which classical fields in the Lagrangian develop a singularity near Σ together with the type of singularity.

A complementary viewpoint on surface operators is to add new degrees of freedom on the surface Σ. Such an approach to surface operators in $\mathcal{N} = 4$ SYM has been considered in [22, 23] where the new degrees of freedom arise from localized open strings on a brane intersection.

The basic effect of $\mathcal{O}_\Sigma$ is to generate an Aharonov-Bohm phase corresponding to a group element U (2.1). If we let z be the complex coordinate in the $R^2 \subset R^{1,3}$ plane normal to Σ, the singularity in the gauge field configuration is then given by

$$A_z = \sum_{I=1}^{k} \frac{A_I}{z^I}, \tag{2.5}$$

where A_I are constant matrices. Scale invariance of the singularity — which we are going to impose — restricts $A_I = 0$ for $I \geq 2$.

The operator $\mathcal{O}_\Sigma$ can also excite a complex scalar field Φ of $\mathcal{N} = 4$ SYM near Σ while preserving half of the Poincare supersymmetries of $\mathcal{N} = 4$ SYM. Imposing that the singularity is scale invariant[9] yields

$$\Phi = \frac{\Phi_1}{z}, \tag{2.6}$$

where Φ_1 is a constant matrix.

A surface operator $\mathcal{O}_\Sigma$ is characterized by the choice of an unbroken gauge group $L \subset G$ along Σ. Correspondingly, the singularity of all the fields excited by $\mathcal{O}_\Sigma$ must be invariant under the unbroken gauge group L. For $L = \prod_{l=1}^{M} \mathrm{U}(N_l) \subset \mathrm{U}(N)$ the singularity in the gauge field is the non-abelian vortex configuration in (2.2) and the two dimensional θ-angles are given by (2.4). L-invariance together with scale invariance requires that Φ develops an L-invariant pole near Σ:

$$\Phi = \frac{1}{z} \begin{pmatrix} \beta_1 + i\gamma_1 \otimes 1_{N_1} & 0 & \ldots & 0 \\ 0 & \beta_2 + i\gamma_2 \otimes 1_{N_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \beta_M + i\gamma_M \otimes 1_{N_M} \end{pmatrix}. \tag{2.7}$$

Therefore, a half-BPS surface operator $\mathcal{O}_\Sigma$ in $\mathcal{N} = 4$ SYM with Levi group $L = \prod_{l=1}^{M} \mathrm{U}(N_l)$ is labeled by $4M$ L-invariant parameters $(\alpha_l, \beta_l, \gamma_l, \eta_l)$ up to the action of S_M, which per-

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[8]For pure gauge theory, the field configuration in (2.2) does satisfy the Yang-Mills equation of motion $D_m F^{mn} = 0$ away from Σ. Moreover, adding the two dimensional θ-angles (2.3) does not change the equations of motion.

[9]If we relax the restriction of scale invariance, one can then get other supersymmetric singularities with higher order poles $\Phi = \sum_{I=1}^{k} \frac{\Phi_I}{z^I}$ and A (2.5). The surface operators associated with these singularities may be relevant [4] for the gauge theory approach to the study of the geometric Langlands program with wild ramification.

mutes the different eigenvalues in (2.2) (2.4) (2.7). The operator is defined by the path integral of $\mathcal{N} = 4$ SYM with the insertion of the operator (2.3) expanded around the L-invariant singularities (2.2) (2.7) and by integrating over smooth fields near Σ. As in the pure gauge theory case, we must mode out by gauge transformations that take values in $L \subset \mathrm{U}(N)$ when restricted to Σ. The surface operator $\mathcal{O}_\Sigma$ becomes singular whenever the the parameters that label the surface operator $(\alpha_l, \beta_l, \gamma_l, \eta_l)$ for $l = 1, \ldots, M$ are such that they are invariant under a larger symmetry than L, the group of gauge transformations we have to mode out when evaluating the path integral.

S-duality of surface operators. In $\mathcal{N} = 4$ SYM the coupling constant combines with the four dimensional θ-angle into a complex parameter taking values in the upper half-plane:

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}. \tag{2.8}$$

The group of duality symmetries of $\mathcal{N} = 4$ SYM is an infinite discrete subgroup of $\mathrm{SL}(2, R)$, which depends on the gauge group G. For $\mathcal{N} = 4$ SYM with $G = \mathrm{U}(N)$ the relevant symmetry group is $\mathrm{SL}(2, Z)$:

$$\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, Z). \tag{2.9}$$

Under S-duality $\tau \to -1/\tau$ and G gets mapped[10] to the S-dual or Langlands dual gauge group $^L G$. For $G = \mathrm{U}(N)$ the S-dual group is $^L G = \mathrm{U}(N)$, and $\mathrm{SL}(2, Z)$ is a symmetry of the theory, which acts on the coupling of the theory by fractional linear transformations:

$$\tau \to \frac{a\tau + b}{c\tau + d}. \tag{2.10}$$

In [4], Gukov and Witten made a proposal of how S-duality acts on the parameters $(\alpha_l, \beta_l, \gamma_l, \eta_l)$ labeling a half-BPS surface operator. The proposed action is given by [4]:

$$\begin{aligned} (\beta_l, \gamma_l) &\to |c\tau + d| \, (\beta_l, \gamma_l) \\ (\alpha_l, \eta_l) &\to (\alpha_l, \eta_l) \mathcal{M}^{-1}. \end{aligned} \tag{2.11}$$

With the aid of this proposal, it was shown in [4] that the gauge theory approach to the geometric Langlands program pioneered in [17] naturally extends to the geometric Langlands program with tame ramification.

Symmetries of half-BPS surface operators in $\mathcal{N} = 4$ SYM. We now describe the unbroken symmetries of the half-BPS surface operators $\mathcal{O}_\Sigma$. These symmetries play an important role in determining the gravitational dual description of these operators, which we provide in the next section.

In the absence of any insertions, $\mathcal{N} = 4$ SYM is invariant under the $PSU(2, 2|4)$ symmetry group. If we consider the surface $\Sigma = R^{1,1} \subset R^{1,3}$, then Σ breaks the $SO(2, 4)$

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[10]For G not a simply-laced group, $\tau \to -1/n\tau$, where n is the ratio of the length-squared of the long and short roots of G.

conformal group to a subgroup. A surface operator $\mathcal{O}_\Sigma$ supported on this surface inserts into the gauge theory a static probe string. This surface is manifestly invariant under rotations and translations in Σ and scale transformations. It is also invariant under the action of inversion $I : x^\mu \to x^\mu/x^2$ and consequently[11] invariant under special conformal transformations in Σ. Therefore, the symmetries left unbroken by $\Sigma = R^{1,1}$ generate an $SO(2,2) \times SO(2)_{23}$ subgroup of the $SO(2,4)$ conformal group, where $SO(2)_{23}$ rotates the plane transverse to Σ in $R^{1,3}$. In Euclidean signature, the surface $\Sigma = S^2$ preserves an $SO(1,3) \times SO(2)_{23}$ subgroup of the Euclidian conformal group. This surface can be obtained from the surface $\Sigma = R^2 \in R^4$ by the action of a broken special conformal generator and can also be used to construct a half-BPS surface operator $\mathcal{O}_\Sigma$ in $\mathcal{N} = 4$ SYM.

Since the symmetry of a surface operator with $\Sigma = R^{1,1}$ is $SO(2,2) \times SO(2)_{23}$ one can study such an operator either by considering the gauge theory in $R^{1,3}$ or in $AdS_3 \times S^1$, which can be obtained from $R^{1,3}$ by a conformal transformation. Studying the gauge theory in $AdS_3 \times S^1$ has the advantage of making the symmetries of the surface operator manifest, as the conformal symmetries left unbroken by the surface act by isometries on $AdS_3 \times S^1$. Surface operators in $R^{1,3}$ are described by a codimension two singularity while surface operators in $AdS_3 \times S^1$ are described by a boundary condition on the boundary of AdS_3. A surface operator with $\Sigma = R^{1,1}$ corresponds to a boundary condition on AdS_3 in Poincare coordinates while a surface operator on $\Sigma = S^2$ corresponds to a boundary condition on global Euclidian AdS_3.

The singularity in the classical fields produced by $\mathcal{O}_\Sigma$ in (2.2) (2.7) is also invariant under $SO(2,2)$. The $\mathcal{N} = 4$ scalar field Φ carries charge under an $SO(2)_R$ subgroup of the $SO(6)$ R-symmetry and is therefore $SO(4)$ invariant. The surface operator $\mathcal{O}_\Sigma$ is therefore invariant under $SO(2,2) \times SO(2)_a \times SO(4)$, where $SO(2)_a$ is generated by the anti-diagonal product[12] of $SO(2)_{23} \times SO(2)_R$.

$\mathcal{N} = 4$ SYM has sixteen Poincare supersymmetries and sixteen conformal supersymmetries, generated by ten dimensional Majorana-Weyl spinors ϵ_1 and ϵ_2 of opposite chirality. As shown in the appendix A, the surface operator $\mathcal{O}_\Sigma$ for $\Sigma = R^{1,1}$ preserves half of the Poincare and half of the conformal supesymmetries[13] and is therefore half-BPS.

With the aid of these symmetries we study in the next section the gravitational description of half-BPS surface operators in $\mathcal{N} = 4$ SYM.

Scaling weight of half-BPS surface operators in $\mathcal{N} = 4$ SYM. Conformal symmetry constraints the form of the OPE of the energy-energy tensor T_{mn} with the operators in the theory. For a surface operator $\mathcal{O}_\Sigma$ supported on $\Sigma = R^{1,1}$, $SO(2,2) \times SO(2)_{23}$ invariance completely fixes the OPE of T_{mn} with $\mathcal{O}_\Sigma$:

$$\frac{< T_{\mu\nu}(x)\mathcal{O}_\Sigma >}{< \mathcal{O}_\Sigma >} = h\frac{\eta_{\mu\nu}}{r^4}; \qquad \frac{< T_{ij}(x)\mathcal{O}_\Sigma >}{< \mathcal{O}_\Sigma >} = \frac{h}{r^4}\left[4n_i n_j - 3\delta_{ij}\right]; \qquad < T_{\mu i}(x)\mathcal{O}_\Sigma >= 0.$$
(2.12)

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[11]We recall that a special conformal transformation K_μ is generated by $IP_\mu I$, where P_μ is the translation generator and I is an inversion.

[12]Since $SO(2)_a$ leaves $\Phi \cdot z$ in (2.7) invariant.

[13]For $\Sigma = S^2$, the operator is also half-BPS, but it preserves a linear combination of Poincare and special conformal supersymmetries.

Here $x^m = (x^\mu, x^i)$, where x^μ are coordinates along Σ and $n^i = x^i/r$, and r is the radial coordinate in the R^2 transverse to $R^{1,1}$. h is the scaling weight of $\mathcal{O}_\Sigma$, which generalizes [24] the notion of conformal dimension of local conformal fields to surface operators.

In order to calculate the scaling dimension of a half-BPS surface operator $\mathcal{O}_\Sigma$ in $\mathcal{N} = 4$ SYM we evaluate the classical field configuration (2.2) (2.4) (2.7) characterizing a half-BPS surface operator on the classical energy-momentum tensor of $\mathcal{N} = 4$ SYM:

$$T_{mn} = \frac{1}{g^2}\text{Tr}\left[D_m\phi D_n\phi - \frac{1}{2}\delta_{mn}(D\phi)^2 - \frac{1}{6}(D_mD_n - \delta_{mn}D^2)\phi^2\right]$$
$$+ \frac{1}{g^2}\text{Tr}\left[-F_{ml}F_{nl} + \frac{1}{4}\delta_{mn}F_{lp}F_{lp}\right]. \tag{2.13}$$

A straightforward computation[14] leads to:

$$h = -\frac{16}{3g^2}\sum_{i=1}^{M}(\beta_i^2 + \gamma_i^2) = -\frac{4}{3}\,\text{Im}\tau\sum_{i=1}^{M}(\beta_i^2 + \gamma_i^2). \tag{2.14}$$

The action of an SL$(2, Z)$ transformation (2.10) on the coupling constant of $\mathcal{N} = 4$ SYM implies that:

$$\text{Im}\tau \rightarrow \frac{\text{Im}\tau}{|c\tau + d|^2}. \tag{2.15}$$

Combining this with the action (2.11) of Montonen-Olive duality on the parameters of the surface operator, we find that the scaling weight (2.14) of a half-BPS surface operator $\mathcal{O}_\Sigma$ is invariant under S-duality:

$$h \rightarrow h. \tag{2.16}$$

In this respect half-BPS surface operators behave like the half-BPS local operators of $\mathcal{N} = 4$ SYM, whose conformal dimension is invariant under SL$(2, Z)$, and unlike the half-BPS Wilson-'t Hooft operators whose scaling weight is not S-duality invariant [24].

3. Bubbling surface operators

In this section we find the dual gravitational description of the half-BPS surface operators $\mathcal{O}_\Sigma$ described in the previous section. The bulk description is given in terms of asymptotically AdS$_5\times$S^5 and singularity free solutions of the Type IIB supergravity equations of motion. The data from which the solution is uniquely determined encodes the corresponding data about the surface operator $\mathcal{O}_\Sigma$.

The strategy to obtain these solutions is to make an ansatz for Type IIB supergravity which is invariant under all the symmetries preserved by the half-BPS surface operators $\mathcal{O}_\Sigma$. As discussed in the previous section, the bosonic symmetries preserved by a half-BPS surface operator $\mathcal{O}_\Sigma$ are SO$(2,2) \times$ SO$(4) \times$ SO$(2)_a$. Therefore the most general ten dimensional metric invariant under these symmetries can be constructed by fibering

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[14]Contact terms depending on α, β, γ and proportional to the derivative of the two-dimensional δ-function appear when evaluating the on-shell energy-momentum tensor. It would be interesting to understand the physical content of these contact terms.

AdS$_3 \times$S$^3 \times$S^1 over a three manifold X, where the symmetries act by isometries on the fiber. The constraints imposed by unbroken supersymmetry on the ansatz are obtained by demanding that the ansatz for the supergravity background possesses a sixteen component Killing spinor, which means that the background solves the Killing spinor equations of Type IIB supergravity. A solution of the Killing spinor equations and the Bianchi identity for the five-form field strength guarantee that the full set of equations of Type IIB supergravity are satisfied and that a half-BPS solution has been obtained.

The problem of solving the Killing spinor equations of Type IIB supergravity with an $SO(2,2) \times SO(4) \times SO(2)_a$ symmetry can be obtained by analytic continuation of the equations studied by LLM [13, 16] , which found the supergravity solutions describing the half-BPS local operators of $\mathcal{N} = 4$ SYM, which have an $SO(4) \times SO(4) \times R$ symmetry. The equations determining the metric and five-form flux can be read from [13, 16], in which the analytic continuation that we need to construct the gravitational description of half-BPS surface operators $\mathcal{O}_\Sigma$ was considered.

The ten dimensional metric and five-form flux is completely determined in terms of data that needs to be specified on the three manifold X in the ten dimensional space. An asymptotically AdS$_5 \times$S^5 metric is uniquely determined in terms of a function $z(x_1, x_2, y)$, where $(x_1, x_2, y) \equiv (\vec{x}, y)$ are coordinates in X. The ten dimensional metric in the Einstein frame is given by[15]

$$ds^2 = y\sqrt{\frac{2z+1}{2z-1}}ds^2_{AdS_3} + y\sqrt{\frac{2z-1}{2z+1}}d\Omega_3 + \frac{2y}{\sqrt{4z^2-1}}(d\chi + V)^2 + \frac{\sqrt{4z^2-1}}{2y}(dy^2 + dx_i dx_i),$$
$$(3.1)$$

where $ds^2_X = dy^2 + dx_i dx_i$ with $y \geq 0$ and V is a one-form in X satisfying $dV = 1/y *_X dz$. AdS$_3$ in Poincare coordinate corresponds to a surface operator on $\Sigma = R^{1,1}$ while AdS$_3$ in global Euclidian coordinate corresponds to a surface operator on $\Sigma =$S^2. The U(1)$_a$ symmetry acts by shifts on χ while $SO(2,2)$ and $SO(4)$ act by isometries on the coordinates of AdS$_3$ and S^3 respectively.

A non-trivial solution to the equations of motion is obtained by specifying a configuration of M point-like particles in X. The data from which the solution is determined is the "charge" Q_l of the particles together with their positions $(\vec{x}_l, y_l)$ in X (see figure 1). Given a "charge" distribution, the function $z(x_1, x_2, y)$ solves the following differential equation:

$$\partial_i \partial_i z(x_1, x_2, y) + y \partial_y \left(\frac{\partial_y z(x_1, x_2, y)}{y} \right) = -\sum_{l=1}^{M} Q_l \delta(y - y_l) \delta^{(2)}(\vec{x} - \vec{x}_l). \qquad (3.2)$$

Introducing a "charge" at the point $(\vec{x}_l, y_l)$ in X has the effect of shrinking[16] the S^1 with coordinate χ in (3.1) to zero size at that point. In order for this to occur in a smooth fashion the magnitude of the "charge" has to be fixed [13, 16] so that $Q_l = 2\pi y_l$. Therefore,

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[15]The "analytic" continuation from the bubbling geometries dual to the half-BPS local operators is given by $z \to z$, $t \to \chi$, $y \to -iy$, $\vec{x} \to i\vec{x}$, $d\Omega_3 \to -ds^2_{AdS_3}$ [13, 16].

[16]Near $y = y_l$ the form of the relevant part of the metric is that of the Taub-NUT space. Fixing the value of the "charge" at $y = y_l$ by imposing regularity of the metric coincides with the usual regularity constraint on the periodicity of the circle in Taub-NUT space.
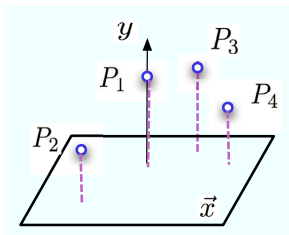
Figure 2: A topologically non-trivial S^5 can be constructed by fibering $S^1 \times S^3$ over an interval connecting the $y = 0$ plane and the location of the "charge" at the point $P_l \in X$ with $(\vec{x}_l, y_l)$ coordinates.

the independent data characterizing the metric and five-form of the solution is the position of the M "charges", given by $(\vec{x}_l, y_l)$.

In summary, a smooth half-BPS $SO(2,2) \times SO(4) \times SO(2)_a$ invariant asymptotically $AdS_5 \times S^5$ metric (3.1) solving the Type IIB supergravity equations of motion is found by solving (3.2) subject to the boundary condition $z(x_1, x_2, 0) = 1/2$ [13, 16], so that the S^3 in (3.1) shrinks in a smooth way at $y = 0$. The function $z(x_1, x_2, y)$ is given by

$$z(x_1, x_2, y) = \frac{1}{2} + \sum_{l=1}^{M} z_l(x_1, x_2, y), \tag{3.3}$$

where

$$z_l(x_1, x_2, y) = \frac{(\vec{x} - \vec{x}_l)^2 + y^2 + y_l^2}{2\sqrt{((\vec{x} - \vec{x}_l)^2 + y^2 + y_l^2)^2 - 4y_l^2 y^2}} - \frac{1}{2}, \tag{3.4}$$

and V can be computed from $z(x_1, x_2, y)$ from $dV = 1/y *_X dz$. Both the metric and five-form field strength are determined by an integer M and by $(\vec{x}_l, y_l)$ for $l = 1, \ldots, M$.

Topology of bubbling solutions and two-form holonomies. The asymptotically $AdS_5 \times S^5$ solutions constructed from (3.3) (3.4) are topologically quite rich. In particular, a solution with M point "charges" has M topologically non-trivial S^5's. We can associate to each point $P_l \in X$ a corresponding five-sphere S_l^5. S_l^5 can be constructed by fibering the $S^1 \times S^3$ in the geometry (3.1) over a straight line between the point $(\vec{x}_l, 0)$ and the point $(\vec{x}_l, y_l)$ in X. The topology of this manifold is indeed an S^5, as an S^5 can be represented[17] by an $S^1 \times S^3$ fibration over an interval where the S^1 and S^3 shrink to zero size at opposite ends of the interval, which is what happens in our geometry where the S^3 shrinks at $(\vec{x}_l, 0)$ while the S^1 shrinks at the other endpoint $(\vec{x}_l, y_l)$.

Following [13, 16] we can now integrate the five-form flux over the topologically non-trivial S^5's (see appendix B):

$$-\frac{1}{\pi^3} \int_{S_l^5} F_5 = y_l^2. \tag{3.5}$$

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[17]This can be seen explicitly by writing $d\Omega_5 = \cos^2 \theta d\Omega_3 + d\theta^2 + \sin^2 \theta d\phi^2$.
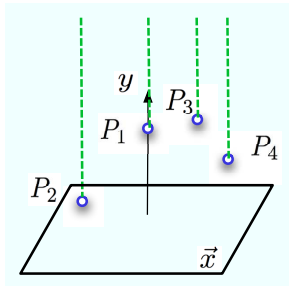
Figure 3: A disk D can be constructed by fibering S^1 over an interval connecting the "charge" at the point $P_l \in X$ with $(\vec{x}_l, y_l)$ coordinates and the boundary of $AdS_5 \times S^5$.

Since flux has to be quantized, the position in the y-axis of the l-th particle in X is also quantized

$$y_l^2 = 4\pi N_l l_p^4 \qquad N_l \in Z, \tag{3.6}$$

where l_p is the ten dimensional Planck length. For an asymptotically $AdS_5 \times S^5$ geometry with radius of curvature $R^4 = 4\pi N l_p^4$, which is dual to $\mathcal{N} = 4$ $U(N)$ SYM, we have that the total amount of five-form flux must be N:

$$N = \sum_{l=1}^{M} N_l. \tag{3.7}$$

The asymptotically $AdS_5 \times S^5$ solutions constructed from (3.3) (3.4) also contain non-trivial surfaces. In particular, a solution with M point "charges" has M non-trivial disks D_l. Just as in the case of the S^5's, we can associate to each point $P_l \in X$ a disk D_l.

Inspection of the asymptotic form of the metric (3.1) given in (3.3) (3.4) reveals that the metric is conformal to $AdS_3 \times S^1$. This geometry on the boundary of $AdS_5 \times S^5$, which is where the dual $\mathcal{N} = 4$ $U(N)$ SYM lives, is the natural background geometry on which to study conformally invariant surface operators in $\mathcal{N} = 4$ SYM. As explained in section 2, an $SO(2, 2) \times SO(2)_{23}$ invariant surface operator can be defined by specifying a codimension two singularity in $R^{1,3}$ or by specifying appropriate boundary conditions for the classical fields in the gauge theory at the boundary of $AdS_3 \times S^1$. In the latter formulation, the worldsheet of the surface operator Σ is the boundary of AdS_3.

Therefore, in the boundary of $AdS_5 \times S^5$ we have a non-contractible S^1. If we fiber the S^1 parametrized by χ in (3.1) over a straight line connecting a point $(\vec{x}_l, y_l)$ in X — where the S^1 shrinks to zero size — to a point in X corresponding to the boundary of $AdS_5 \times S^5$ — given by $\vec{x}, y \to \infty$ — we obtain a surface D_l. This surface is topologically a disk[18] and there are M of them for a "charge" distribution of M particles in X.

Due to the existence of the disks D_i, the supergravity solution given by the metric and five-form flux alone is not unique. Type IIB supergravity has a two-form gauge field from the NS-NS sector and another one from the RR sector. In order to fully specify a solution

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[18]Such disks also appear in the study of the high temperature regime of $\mathcal{N} = 4$ SYM, where the bulk geometry [25] is the AdS Schwarzschild black hole, which also has a non-contractible S^1 in the boundary which is contractible in the full geometry.
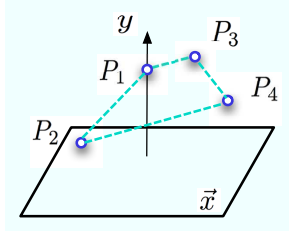
Figure 4: An S^2 can be constructed by fibering S^1 over an interval connecting the "charge" at the point $P_l \in X$ with a different "charge" at point $P_m \in X$.

of Type IIB supergravity in the bubbling geometry (3.1) we must complement the metric and the five-form with the integral of the two-forms around the disks[19]

$$\alpha_l = -\int_{D_l} \frac{B_{NS}}{2\pi} \qquad\qquad \eta_l = \int_{D_l} \frac{B_R}{2\pi} \qquad l = 1, \dots, M, \qquad (3.8)$$

where we have used notation conducive to the later comparison with the parameters characterizing a half-BPS surface operator $\mathcal{O}_\Sigma$. Since both B_{NS} and B_R are invariant under large gauge transformations, the parameters (α_l, η_l) take values on a circle of unit radius.

Apart from the M disks D_l, the bubbling geometry constructed from (3.3) (3.4) also has topologically non-trivial S^2's. One can construct an S^2 by fibering the S^1 in (3.1) over a straight line connecting the points P_l and P_m in X. Since the S^1 shrinks to zero size in a smooth manner at the endpoints we obtain an S^2. Therefore, to every pair of "charges" in X, characterized by different points P_l and P_m in X, we can construct a corresponding S^2, which we label by $S^2_{l,m}$. The integral of B_{NS} and B_R over $S^2_{l,m}$ do not give rise to new parameters, since $[S^2_{l,m}] = [D_l] - [D_m]$ in homology, and the periods can be determined from (3.8).

Bubbling geometries as surface operators. As we discussed in section 2, a surface operator $\mathcal{O}_\Sigma$ is characterized by an unbroken gauge group $L \in U(N)$ along together with $4M$ L-invariant parameters $(\alpha_l, \beta_l, \gamma_l, \eta_l)$. On the other hand, the Type IIB supergravity solutions we have described depend on the positions $(\vec{x}_l, y_l)$ of M "charged" particles in X and the two-form holonomies:

$$\int_{D_l} \frac{B_{NS}}{2\pi} \qquad\qquad \int_{D_l} \frac{B_R}{2\pi}. \qquad (3.9)$$

We now establish an explicit dictionary between the parameters in gauge theory and the parameters in supergravity.

For illustration purposes, it is convenient to start by considering the half-BPS surface operator $\mathcal{O}_\Sigma$ with the largest Levi group L, which is $L = U(N)$ for $G = U(N)$. $U(N)$ invariance requires that the singularity in the fields produced by $\mathcal{O}_\Sigma$ take values in the

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[19]The overall signs in the identification are fixed by demanding consistent action of S-duality of $\mathcal{N} = 4$ SYM with that of Type IIB supergravity.

center of $\mathrm{U}(N)$. Therefore, the gauge field and scalar field produced by $\mathcal{O}_\Sigma$ is given by

$$A = \alpha_0 1_N d\theta$$
$$\Phi = \frac{1}{z}(\beta_0 + i\gamma_0)1_N, \tag{3.10}$$

where 1_N is the identity matrix. We can also turn a two-dimensional θ-angle (2.3) for the overall $\mathrm{U}(1)$, so that:

$$\eta = \eta_0 1_N. \tag{3.11}$$

We now identify this operator with the supergravity solution obtained by having a single point "charge" source in X (see figure 1b). If we let the position of the "charge" be $(\vec{x}_0, y_0)$ then

$$z(x_1, x_2, y) = \frac{(\vec{x} - \vec{x}_0)^2 + y^2 + y_0^2}{2\sqrt{((\vec{x} - \vec{x}_0)^2 + y^2 + y_0^2)^2 - 4y_0^2 y^2}}$$
$$V_I = -\epsilon_{IJ}\frac{(x^J - x_0^J)((\vec{x} - \vec{x}_0)^2 + y^2 - y_0^2)}{2(\vec{x} - \vec{x}_0)^2\sqrt{((\vec{x} - \vec{x}_0)^2 + y^2 + y_0^2)^2 - 4y_0^2 y^2}}, \tag{3.12}$$

where $V = V_I dx^I$. The metric (3.1) obtained using (3.12) is the metric of $\mathrm{AdS}_5 \times \mathrm{S}^5$. This can be seen by the following change of variables [16]

$$x^1 - x_0^1 + i(x^2 - x_0^2) = re^{i(\phi - \psi)}$$
$$r = y_0 \sinh u \sin \theta$$
$$y = y_0 \cosh u \cos \theta$$
$$\chi = \frac{1}{2}(\psi + \phi), \tag{3.13}$$

which yields the $\mathrm{AdS}_5 \times \mathrm{S}^5$ metric with AdS_5 foliated by $\mathrm{AdS}_3 \times \mathrm{S}^1$ slices:

$$ds^2 = y_0\left[(\cosh^2 u\, ds_{AdS_3}^2 + du^2 + \sinh^2 u\, d\psi^2) + (\cos^2\theta d\Omega_3 + d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{3.14}$$

We note that the $\mathrm{U}(1)_a$ symmetry of the metric (3.1) — which acts by shifts on χ — identifies via (3.13) an $\mathrm{SO}(2)_R$ subgroup of the the $\mathrm{SO}(6)$ symmetry of the S^5, acting by shifts on ϕ, with an $\mathrm{SO}(2)_{23}$ subgroup of the $\mathrm{SO}(2,4)$ isometry group of AdS_5, acting by opposite shifts on ψ. This is precisely the same combination of generators discussed in section 2 that is preserved by a half-BPS surface operator $\mathcal{O}_\Sigma$ in $\mathcal{N} = 4$ SYM.

The radius of curvature of $\mathrm{AdS}_5 \times \mathrm{S}^5$ in (3.14) is given by $R^4 = y_0^2$. Therefore using that $R^4 = 4\pi N l_p^4$, where N is the rank of the $\mathcal{N} = 4$ YM theory, we have that

$$N = \frac{y_0^2}{4\pi l_p^4}, \tag{3.15}$$

and the position of the "charge" in y gets identified with the rank of the unbroken gauge group and is therefore quantized.

The residue of the pole in Φ (3.10) gets identified with the position of the "charge" in the $\vec{x}$-plane. It follows from (3.1) that the coordinates $\vec{x}$ and y have dimensions of length^2.

Therefore, we identify the residue of the pole of Φ with the position of the "charge" in the $\vec{x}$-plane in X via:

$$(\beta_0, \gamma_0) = \frac{\vec{x}_0}{l_s^2}. \tag{3.16}$$

Unlike the position in y, the position in $\vec{x}$ is not quantized.

The remaining parameters of the surface operator $\mathcal{O}_\Sigma$ with $\mathrm{U}(N)$ Levi group — given by (α_0, η_0) — get identified with the holonomy of the two-forms of Type IIB supergravity over D

$$\alpha_0 = -\int_D \frac{B_{NS}}{2\pi} \qquad \eta_0 = \int_D \frac{B_R}{2\pi}, \tag{3.17}$$

where D is the disk ending on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ boundary on the S^1. This identification properly accounts for the correct periodicity of these parameters, which take values on a circle of unit radius.

The path integral which defines a half-BPS surface operator $\mathcal{O}_\Sigma$ when $L = \mathrm{U}(N)$ is never singular as the gauge symmetry cannot be further enhanced by changing the parameters $(\alpha_0, \beta_0, \gamma_0, \eta_0)$ of the surface operator. Correspondingly, the dual supergravity solution with one "charge" also never acquires a singularity by changing the parameters of the solution.

Let's now consider the most general half-BPS surface operator $\mathcal{O}_\Sigma$. First we need to characterize the operator by its Levi group, which for a $\mathrm{U}(N)$ gauge group takes the form $\prod_{l=1}^M U(N_l)$ with $N = \sum_{l=1}^M N_l = N$. The operator then depends on $4M$ L-invariant parameters $(\alpha_l, \beta_l, \gamma_l, \eta_l)$ for $l = 1 \ldots, M$ up to the action of S_M, which acts by permuting the parameters.

The corresponding supergravity solution associated to such an operator is given by the metric (3.1). The number of unbroken gauge group factors — given by the integer M — corresponds to the number of point "charges" in (3.2). For $M > 1$, the metric that follows from (3.3) (3.4) is $\mathrm{AdS}_5 \times \mathrm{S}^5$ only asymptotically and not globally.

The rank of the various gauge group factors in the Levi group $\prod_{l=1}^M U(N_l)$ — given by the integers N_l — correspond to the position of the "charges" along $y \in X$, given by the coordinates y_l. The precise identification follows from (3.5) (3.6):

$$N_l = \frac{y_l^2}{4\pi l_p^4} \qquad l = 1, \ldots, M. \tag{3.18}$$

N_l also corresponds to the amount of five-form flux over S_l^5, the S^5 associated with the l-th point charge:

$$N_l = -\frac{1}{4\pi^4 l_p^4} \int_{S_l^5} F_5 \qquad l = 1, \ldots, M. \tag{3.19}$$

This identification quantizes the y coordinate in X into l_p size bits. Thus length is quantized as opposed to area, which is what happens for the geometry dual to the half-BPS local operators [13, 16], where it can be interpreted as the quantization of phase space in the boundary gauge theory.

A half-BPS surface operator $\mathcal{O}_\Sigma$ develops a pole for the scalar field Φ (2.7). The pole is characterized by its residue, which is given by $2M$ real parameters (β_l, γ_l). These

parameters are identified with the position of the M "charges" in the $\vec{x}$-plane in X via:

$$(\beta_l, \gamma_l) = \frac{\vec{x}_l}{l_s^2}.\tag{3.20}$$

All these parameters take values on the real line.

The remaining parameters characterizing a half-BPS surface operator $\mathcal{O}_\Sigma$ are the periodic variables (α_l, η_l), which determine the holonomy produced by $\mathcal{O}_\Sigma$ and the corresponding two-dimensional θ-angles. These parameters get identified with the holonomy of the two-forms of Type IIB supergravity over the M non-trivial disks D_l that the geometry generates in the presence of M "charges" in X:

$$\begin{aligned}
\alpha_l &= -\int_{D_l} \frac{B_{NS}}{2\pi} \\
\eta_l &= \int_{D_l} \frac{B_R}{2\pi}.
\end{aligned}\tag{3.21}$$

The identification respects the periodicity of (α_l, η_l), which in supergravity arises from the invariance of B_{NS} and B_R under large gauge transformations.[20]

We have given a complete dictionary between all the parameters that a half-BPS surface operator in $\mathcal{N} = 4$ SYM depends on and all the parameters in the corresponding bubbling geometry. We note that a surface operator $\mathcal{O}_\Sigma$ depends on a set of parameters up to the action of the permutation group S_M on the parameters, which is part of the U(N) gauge symmetry. The corresponding statement in supergravity is that the solution dual to a surface operator is invariant under the action of S_M, which acts by permuting the "charges" in X.

The supergravity solution is regular as long as the "charges" do not collide. A singularity arises whenever two point "charges" in X coincide (see figure 1):

$$(\vec{x}_l, y_l) \rightarrow (\vec{x}_m, y_m) \qquad \text{for } l \neq m.\tag{3.22}$$

Whenever this occurs, there is a reduction in the number of independent disks since (see figure 3):

$$D_l \rightarrow D_m \qquad \text{for } l \neq m,\tag{3.23}$$

and therefore

$$\begin{aligned}
\int_{D_l} \frac{B_{NS}}{2\pi} &\rightarrow \int_{D_m} \frac{B_{NS}}{2\pi} \\
\int_{D_l} \frac{B_R}{2\pi} &\rightarrow \int_{D_m} \frac{B_R}{2\pi} \qquad \text{for } l \neq m.
\end{aligned}\tag{3.24}$$

In this limit of parameter space the non-trivial S^2 connecting the points P_l and P_m in X shrinks to zero size as $[S^2_{l,m}] = [D_l] - [D_m] \rightarrow 0$, and the geometry becomes singular.

---

[20]Since the gauge invariant variables are $e^{i \int_D \frac{B}{2\pi}}$.

By using the dictionary developed in this paper, such a singular geometry corresponds to a limit when two of each of the set of parameters $(\alpha_l, \beta_l, \gamma_l, \eta_l)$ defining a half-BPS surface operator $\mathcal{O}_\Sigma$ become equal:

$$\alpha_l \to \alpha_m, \beta_l \to \beta_m, \gamma_l \to \gamma_m, \eta_l \to \eta_m \qquad \text{for } l \neq m. \tag{3.25}$$

In this limit the unbroken gauge group preserved by the surface operator $\mathcal{O}_\Sigma$ is enhanced to L' from the original Levi group $\prod_{l=1}^{M} U(N_l)$, where $L \subset L'$. As explained in section 2 the path integral from which $\mathcal{O}_\Sigma$ is defined becomes singular.

In summary, we have found the description of all half-BPS surface operators $\mathcal{O}_\Sigma$ in $\mathcal{N} = 4$ SYM in terms of solutions of Type IIB supergravity. The asymptotically $\text{AdS}_5 \times \text{S}^5$ solutions are regular and when they develop a singularity then the corresponding operator also becomes singular.

S-duality of surface operators from type IIB supergravity The group of dualities of $\mathcal{N} = 4$ SYM acts non-trivially [4] on surface operators $\mathcal{O}_\Sigma$ (see discussion in section 2). For $G = U(N)$ the duality group is $\text{SL}(2, Z)$ and its proposed action on the parameters on which $\mathcal{O}_\Sigma$ depends on is [4]:

$$(\beta_l, \gamma_l) \to |c\tau + d| \, (\beta_l, \gamma_l)$$
$$(\alpha_l, \eta_l) \to (\alpha_l, \eta_l)\mathcal{M}^{-1}, \tag{3.26}$$

where $\mathcal{M}$ is an $\text{SL}(2, Z)$ matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.27}$$

We now reproduce[21] this transformation law by studying the action of the $\text{SL}(2, Z)$ subgroup of the $\text{SL}(2, R)$ classical symmetry of Type IIB supergravity, which is in fact the appropriate symmetry group of Type IIB string theory. For that we need to analyze the action of S-duality on our bubbling geometries.

$\text{SL}(2, Z)$ acts on the complex scalar $\tau = C_0 + ie^{-\phi}$ of Type IIB supergravity in the familiar fashion

$$\tau \to \frac{a\tau + b}{c\tau + d}, \tag{3.28}$$

where as usual τ gets identified with the complexified coupling constant of $\mathcal{N} = 4$ SYM (2.8). $\text{SL}(2, Z)$ also rotates the two-form gauge fields[22] of Type IIB supergravity

$$\begin{pmatrix} B_{NS} \\ B_R \end{pmatrix} \to \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} B_{NS} \\ B_R \end{pmatrix}, \tag{3.29}$$

while leaving the metric in the Einstein frame and the five-form flux invariant.

Given that the metric in (3.1) is in the Einstein frame, $\text{SL}(2, Z)$ acts trivially on the coordinates $(\vec{x}, y)$. Nevertheless, since

$$l_s = g_s^{-1/4} l_p \qquad \text{with } g_s = e^\phi \tag{3.30}$$

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[21]If we apply the same idea to the LLM geometries dual to half-BPS local operators in [13], we conclude that the half-BPS local operators are invariant under S-duality.

[22]See e.g. [26, 27].

the string scale transforms under $SL(2, Z)$ as follows:

$$l_s^2 \rightarrow \frac{l_s^2}{|c\tau + d|}. \tag{3.31}$$

Therefore, under S-duality:

$$\frac{\vec{x}_l}{l_s^2} \rightarrow |c\tau + d| \, \frac{\vec{x}_l}{l_s^2}. \tag{3.32}$$

Given our dictionary in (3.20), we find that the surface operator parameters (β_l, γ_l) transform as in (3.26), agreeing with the proposal in [4].

The identification of the rest of the parameters is (3.21):

$$\alpha_l = - \int_{D_l} \frac{B_{NS}}{2\pi}$$
$$\eta_l = \int_{D_l} \frac{B_R}{2\pi}. \tag{3.33}$$

Using the action of $SL(2, Z)$ on the two-forms (3.29) and the identification (3.33), it follows from a straightforward manipulation that the surface operator paramaters (α_l, η_l) transform as in (3.26), agreeing with the proposal in [4].

Acknowledgments

A. Supersymmetry of surface operator in $\mathcal{N}{=}4$ SYM

In this appendix we study the Poincare and conformal supersymmetries preserved by a surface operator in $\mathcal{N}{=}4$ SYM supported on $R^{1,1}$. These symmetries are generated by ten dimensional Majorana-Weyl spinors ϵ_1 and ϵ_2 of opposite chirality. We determine the supersymmetries left unbroken by a surface operator by studying the supersymmetry variation of the gaugino in the presence of the surface operator singularity in (2.2) (2.7).

The metric is given by:

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = -(dx^0)^2 + (dx^1)^2 + 2dzd\bar{z}. \tag{A.1}$$

where $z = \frac{1}{\sqrt{2}}(x^2 + ix^3) = |z|e^{i\theta}$, while the singularity in the fields is

$$\Phi^\omega = \Phi_{\bar{\omega}} = \frac{1}{\sqrt{2}}(\Phi^8 + i\Phi^9) = \frac{\beta + i\gamma}{z}$$
$$A = \alpha d\theta, F = dA = 2\pi\alpha\delta_D, \tag{A.2}$$

where $[\alpha, \beta] = [\alpha, \gamma] = [\beta, \gamma] = 0$ and $\delta_D = d(d\theta)$ is a two form delta function. The relevant Γ-matrices are

$$\Gamma_z = \frac{1}{\sqrt{2}}(\Gamma_2 + i\Gamma_3)$$
$$\Gamma_{\bar{z}} = \frac{1}{\sqrt{2}}(\Gamma_2 - i\Gamma_3)$$
$$\{\Gamma_z, \Gamma_{\bar{z}}\} = 2. \tag{A.3}$$

A Poincare supersymmetry transformation is given by

$$\delta\lambda = \left(\frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu} + \nabla_\mu\Phi_i\Gamma^{\mu i} + \frac{i}{2}[\Phi_i, \Phi_j]\Gamma^{ij}\right)\epsilon_1 \tag{A.4}$$

where μ runs from 0 to 3 and i runs from 4 to 9, while a superconformal supersymmetry transformation is given by

$$\delta\lambda = \left[\left(\frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu} + \nabla_\mu\Phi_i\Gamma^{\mu i} + \frac{i}{2}[\Phi_i, \Phi_j]\Gamma^{ij}\right)x^\sigma\Gamma_\sigma - 2\Phi_i\Gamma^i\right]\epsilon_2 \tag{A.5}$$

From (A.4), it follows that the unbroken Poincare supersymmetries are given by:

$$\Gamma^{\bar{z}\omega}\epsilon_1 = 0 \Leftrightarrow \Gamma_{2389}\epsilon_1 = -\epsilon_1. \tag{A.6}$$

The unbroken superconformal supersymmetries are given by:

$$\left[\left(-\frac{\beta + i\gamma}{z^2}\Gamma^{z\bar{\omega}} - \frac{\beta - i\gamma}{\bar{z}^2}\Gamma^{\bar{z}\omega}\right)(x_0\Gamma^0 + x_1\Gamma^1 + z\Gamma^z + \bar{z}\Gamma^z) - 2\Phi_\omega\Gamma^\omega - 2\Phi_{\bar{\omega}}\Gamma^{\bar{\omega}}\right]\epsilon_2 = 0. \tag{A.7}$$

From the terms proportional to x^0 and x^1, we find that the unbroken superconformal supersymmetries are given by:

$$\Gamma^{z\bar{\omega}}\epsilon_2 = 0 \quad \Leftrightarrow \quad \Gamma_{2389}\epsilon_2 = -\epsilon_2. \tag{A.8}$$

The rest of the conditions

$$[\Gamma^{\bar{z}}\Gamma^{\bar{\omega}}\Gamma^z]\epsilon_2 = 0, \tag{A.9}$$

are automatically satisfied once (A.8) is imposed.

We conclude that the singularity (2.2) (2.7) is half-BPS and that the preserved supersymmetry is generated by ϵ_1 and ϵ_2 subject o the constraints $\Gamma_{2389}\epsilon_1 = -\epsilon_1$ and $\Gamma_{2389}\epsilon_2 = -\epsilon_2$.

By acting with a broken special conformal transformation on $\Sigma = R^2 \subset R^4$ to get a surface operator supported on $\Sigma = S^2$, one can show following [28] that such an operator also preserves half of the thirty-two supersymmetries, but are now generated by a linear combination of the Poincare and special conformal supersymmetries.

B. Five form flux

In this appendix, we calculate (3.5) explicitly to evaluate the flux over a non-trivial S^5.

The five-form flux is [13, 16]

$$
F_5 = -\frac{1}{4}\left\{d\left[y^2\frac{2z+1}{2z-1}(d\chi+V)\right] + y^3 *_3 d\left(\frac{z+\frac{1}{2}}{y^2}\right)\right\} \wedge dVol_{AdS_3}
$$
$$
-\frac{1}{4}\left\{d\left[y^2\frac{2z-1}{2z+1}(d\chi+V)\right] + y^3 *_3 d\left(\frac{z-\frac{1}{2}}{y^2}\right)\right\} \wedge d\Omega_3 \tag{B.1}
$$

The five-cycle S^5_l in the bubbling geometry is spanned by coordinates Ω_3, χ and y. Then the integration is:

$$
-\frac{1}{4\pi^4 l_p^4}\int_{S^5_l} F_5 = \frac{1}{16\pi^4 l_p^4}\int d\left[y^2\frac{2z-1}{2z+1}d\chi\right] \wedge d\Omega_3 = \frac{y_l^2}{4\pi l_p^4} = N. \tag{B.2}
$$

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [`hep-th/9711200`].

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys. Lett. **B 428** (1998) 105 [`hep-th/9802109`].

[3] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253 [`hep-th/9802150`].

[4] S. Gukov and E. Witten, *Gauge theory, ramification and the geometric langlands program*, `hep-th/0612073`.

[5] J. Gomis and F. Passerini, *Holographic Wilson loops*, JHEP **08** (2006) 074 [`hep-th/0604007`].

[6] S.-J. Rey and J.-T. Yee, *Macroscopic strings as heavy quarks in large-N gauge theory and anti-de Sitter supergravity*, Eur. Phys. J. **C 22** (2001) 379 [`hep-th/9803001`].

[7] J.M. Maldacena, *Wilson loops in large-N field theories*, Phys. Rev. Lett. **80** (1998) 4859 [`hep-th/9803002`].

[8] N. Drukker and B. Fiol, *All-genus calculation of Wilson loops using D-branes*, JHEP **02** (2005) 010 [`hep-th/0501109`].

[9] S. Yamaguchi, *Bubbling geometries for half BPS Wilson lines*, Int. J. Mod. Phys. **A 22** (2007) 1353 [`hep-th/0601089`].

[10] S. Yamaguchi, *Wilson loops of anti-symmetric representation and d5- branes*, JHEP **05** (2006) 037 [`hep-th/0603208`].

[11] O. Lunin, *On gravitational description of Wilson lines*, JHEP **06** (2006) 026 [`hep-th/0604133`].

[12] J. Gomis and F. Passerini, *Wilson loops as D3-branes*, JHEP **01** (2007) 097 [`hep-th/0612022`].

[13] H. Lin, O. Lunin and J.M. Maldacena, *Bubbling AdS space and 1/2 BPS geometries*, JHEP **10** (2004) 025 [`hep-th/0409174`].

[14] J. Gomis and C. Romelsberger, *Bubbling defect CFT's*, *JHEP* **08** (2006) 050 [hep-th/0604155].

[15] J. Gomis and T. Okuda, *Wilson loops, geometric transitions and bubbling Calabi-Yau's*, *JHEP* **02** (2007) 083 [hep-th/0612190].

[16] H. Lin and J.M. Maldacena, *Fivebranes from gauge theory*, *Phys. Rev.* **D 74** (2006) 084014 [hep-th/0509235].

[17] A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric langlands program*, hep-th/0604151.

[18] E. Frenkel, *Lectures on the langlands program and conformal field theory*, hep-th/0512172.

[19] R.M. Rohm, *Some current problems in particle physics beyond the standard model*, UMI-85-18643, Ph.D. Thesis (1985).

[20] J. Preskill and L.M. Krauss, *Local discrete symmetry and quantum mechanical hair*, *Nucl. Phys.* **B 341** (1990) 50.

[21] M.G. Alford, K.-M. Lee, J. March-Russell and J. Preskill, *Quantum field theory of nonabelian strings and vortices*, *Nucl. Phys.* **B 384** (1992) 251 [hep-th/9112038].

[22] A. Kapustin and S. Sethi, *The Higgs branch of impurity theories*, *Adv. Theor. Math. Phys.* **2** (1998) 571 [hep-th/9804027].

[23] N.R. Constable, J. Erdmenger, Z. Guralnik and I. Kirsch, *Intersecting D3-branes and holography*, *Phys. Rev.* **D 68** (2003) 106007 [hep-th/0211222].

[24] A. Kapustin, *Wilson-'t Hooft operators in four-dimensional gauge theories and S-duality*, *Phys. Rev.* **D 74** (2006) 025005 [hep-th/0501015].

[25] E. Witten, *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [hep-th/9803131].

[26] J. Polchinski, *String theory. Vol. 2*, Chapter 12, Cambridge University Press, U.K. (1998).

[27] K. Becker, M. Becker and J. Schwarz, *String theory and M-theory: a modern introduction*, Chapter 8, Cambridge University Press, U.K. (2007).

[28] M. Bianchi, M.B. Green and S. Kovacs, *Instanton corrections to circular Wilson loops in $N = 4$ supersymmetric Yang-Mills*, *JHEP* **04** (2002) 040 [hep-th/0202003].

[29] J. Gomis, J. Gomis and K. Kamimura, *Non-relativistic superstrings: a new soluble sector of $AdS_5 \times S^5$*, *JHEP* **12** (2005) 024 [hep-th/0507036].

[30] J. Gomis, *Anomaly cancellation in noncritical string theory*, *JHEP* **10** (2005) 095 [hep-th/0508132].

[31] J. Gomis and H. Ooguri, *Non-relativistic closed string theory*, *J. Math. Phys.* **42** (2001) 3127 [hep-th/0009181].